

THE DIFFICULTY OF SYMPLECTIC ANALYSIS WITH SECOND CLASS SYSTEMS

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Abstract

Using the basic concepts of chain by chain method we show that the symplectic analysis, which was claimed to be equivalent to the usual Dirac method, fails when second class constraints are present. We propose a modification in symplectic analysis that solves the problem.

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1 Introduction

There are some attempts to study a constrained system in the framework of first order Lagrangian [1, 2]. The coordinates appearing in a first order Lagrangian are in fact the phase space coordinates. The Euler-Lagrange equations of motion of a first order Lagrangian in an ordinary (non-constrained) system are the same as the canonical equations of motion. The kinetic term in a first order Lagrangian constitutes of a one-form whose exterior derivative appears in the equations of motion. The resulted two-form, called the symplectic tensor, is singular for a constrained system. If the system is not constrained, usually the inverse of the symplectic tensor exists and provides the fundamental Poisson brackets (we exclude degenerate systems discussed in [17, 18] in which the symplectic tensor may have a lower rank in some regions of the phase space).

The properties of a constrained system can be determined by trying to overcome the singularity of the symplectic tensor. Faddeev and Jackiw [3] used the Darboux theorem to separate canonical and non-canonical coordinates. They solved the equations of motion for non-canonical coordinates either to decrease the degrees of singularity of the symplectic tensor or to find the next level constraints.

Then using a special system of coordinates, the authors of [4] showed that the Faddeev-Jackiw approach is essentially equivalent to the usual Dirac method [5]. In a parallel approach, known as symplectic analysis [6, 7, 8, 9] one extends the phase space to include the Lagrange multipliers. In this approach the consistency of constraints at each level adds some additional elements to the symplectic tensor. In other words, the kinetic part of the (first order) Lagrangian is responsible to impose the consistency conditions.

The important point in most papers written in Faddeev-Jackiw method or symplectic analysis is that they often show their results for the constraints in the first level and then *deduce* that the same thing would be repeated at any level. However, the whole procedure of studying the singularities of symplectic tensor, demonstrates some global aspects. For example, some questions that may arise are as follows:

What happens, after all, to the symplectic tensor? Is it ultimately singular? How many degrees of singularity may it have? What is the relation of ultimate singularities with the gauge symmetries of the system? and so on. In [10] we showed that the symplectic analysis gives, at each step, the same

results as the traditional Dirac method (in the framework of *level by level approach*). The symplectic analysis may also be studied in the framework of *chain by chain approach*[11] to obtain the Dirac constraints.

Meanwhile, some recent observation [12] shows that in some examples the result of symplectic analysis and the well-established method of Dirac are not the same. This creates serious doubt about the validity of the symplectic analysis. Therefore, it is worth studying the origin of the difference between this approach and that of Dirac [5]. This is the aim of this paper. In the next section we first review the basic concept of symplectic approach as given in [10]. As we will show the symplectic analysis is equivalent to a special procedure in Dirac approach in which one uses the extended Hamiltonian at each level of consistency. In section(3) we will show that in the framework of Dirac method one is not allowed to use an extended Hamiltonian when there exist second class constraints. The important point to be emphasized is that this result can be understood more clearly in the framework of chain by chain method. In section(4) we show that for a one chain system with second class constraints the symplectic analysis as proposed in the literature fails. This result can be simply generalized to the general case of a multi-chain system. When recognizing the origin of the problem, we give our prescription to solve it in section(5). Finally in section(6) we give an example.

The last point to be noticed is that the problem would not show itself for systems with two levels of constraints. As we will show, this is the case for second class systems with at least four levels of constraints. That is the reason for the fact that the problem does not appear if one considers just first level of constraints.

2 Review of symplectic approach

Consider a phase space with coordinates $y^i (i = 1, \dots, 2K)$ specified by the first order Lagrangian

$$L = a_i(y)\dot{y}^i - H(y) \quad (1)$$

where $H(y)$ is the canonical Hamiltonian of the system. The equations of motion read

$$f_{ij}\dot{y}^j = \partial_i H \quad (2)$$

where $\partial_i \equiv \frac{\partial}{\partial y^i}$ and the presymplectic tensor f_{ij} is defined as

$$f_{ij} \equiv \partial_i a_j(y) - \partial_j a_i(y). \quad (3)$$

We denote it in matrix notation as f . This matrix is invertible for a regular system. Let f^{ij} be the components of the inverse, f^{-1} . From (2) we have

$$\dot{y}^i = \{y^i, H\}, \quad (4)$$

where the Poisson bracket $\{, \}$ is defined as

$$\{F(y), G(y)\} = f^{ij} \partial_i F \partial_j G. \quad (5)$$

If f is singular, then using the Darboux theorem, as shown in [3], one can choose the independent coordinates (y'^α, λ^l) such that

$$L = a'_\alpha \dot{y}'^\alpha - \lambda^l \Phi_l(y') - H(y') \quad (6)$$

where $f'_{\alpha\beta} = \partial_\alpha a'_\beta - \partial_\beta a'_\alpha$ is invertible. This shows that one can consider a system with a singular tensor f_{ij} , as a regular one described by

$$L = a'_\alpha \dot{y}'^\alpha - H(y') \quad (7)$$

together with by the primary constraints $\Phi_l(y')$. In other words, without losing the generality one can assume that one is at first given the first order Lagrangian (1) with a regular presymplectic two-form (3), and then the set of primary constraints $\Phi_\mu^{(1)} (\mu = 1, \dots, M)$ are applied to the system. In this way the system is described by the Lagrangian

$$L = a_i \dot{y}^i - \lambda^\mu \Phi_\mu^{(1)} - H(y) \quad (8)$$

in the extended space (y^i, λ^μ) . The equations of motion (2) should be replaced in matrix form by

$$\left(\begin{array}{c|c} f & 0 \\ \hline 0 & 0 \end{array} \right) \left(\begin{array}{c} \dot{y} \\ \lambda \end{array} \right) = \left(\begin{array}{c} \partial H \\ \Phi^{(1)} \end{array} \right) \quad (9)$$

which is equivalent to Eq. (2) together with the constraint equations $\Phi_\mu^{(1)} = 0$ ($\mu = 1, \dots, M$).

Now one should impose the consistency conditions $\dot{\Phi}_\mu^{(1)} = 0$. To do this, one should extend the space to include new variables η^μ and add the term

$\eta^\mu \dot{\Phi}_\mu^{(1)}$ (or equivalently $-\dot{\eta}^\mu \Phi_\mu^{(1)}$) to the Lagrangian (8). This leads in the extended space (y, λ, η) to the equations

$$\left(\begin{array}{c|c|c} f & 0 & A \\ \hline 0 & 0 & 0 \\ \hline -A & 0 & 0 \end{array} \right) \left(\begin{array}{c} \dot{y} \\ \dot{\lambda} \\ \dot{\eta} \end{array} \right) = \left(\begin{array}{c} \frac{\partial H}{\partial \Phi^{(1)}} \\ 0 \end{array} \right) \quad (10)$$

where the elements of the rectangular matrix A are given by

$$A_{\mu i} = \partial_i \Phi_\mu^{(1)}. \quad (11)$$

However, nothing would be lost if one forgets about the variables λ^μ and reduces the system to the Lagrangian

$$L^{(1)} = a_i \dot{y}^i - \dot{\eta}^\mu \Phi_\mu^{(1)} - H(y). \quad (12)$$

This leads to the symplectic two-form

$$F = \left(\begin{array}{c|c} f & A \\ \hline -A & 0 \end{array} \right) \quad (13)$$

in the $(2K + M)$ dimensional space of variables $Y \equiv (y^i, \eta^\mu)$. It should be noted that the Lagrangian $L^{(1)}$ in Eq. (12) is the same as Eq. (8) in which λ^μ is replaced by $\dot{\eta}^\mu$. This means that the derivatives $\dot{\eta}^\mu$ have the same role as Lagrangian multipliers λ^μ corresponding to primary constraints in the total Hamiltonian

$$H_T = H + \lambda^\mu \Phi_\mu^{(1)}. \quad (14)$$

In other words, if some of $\dot{\eta}^\mu$'s are found by the dynamical equations of the system, then the corresponding Lagrange multipliers are obtained. In Dirac approach [14] this would be the case if there exist some second class constraints.

The equations of motion due to the Lagrangian $L^{(1)}$ can be written in matrix notation as

$$F \dot{Y} = \partial H. \quad (15)$$

Using operations that keep the determinant invariant, it is easy to show that

$$\begin{aligned} \det F &= \det \left(\begin{array}{c|c} f & A \\ \hline 0 & A f^{-1} A \end{array} \right) \\ &= (\det f)(\det \tilde{A} f^{-1} A). \end{aligned} \quad (16)$$

Since $\det f \neq 0$, F would be singular if $C \equiv \tilde{A}f^{-1}A$ is singular. Using (5) and (11) we have

$$C_{\mu\nu} = \left\{ \Phi_{\mu}^{(1)}, \Phi_{\nu}^{(1)} \right\}. \quad (17)$$

Suppose $\text{rank}(C) = M''$ where $M'' \leq M$. This means that F possesses $M' = M - M''$ null-eigenvectors. One can, in principle, divide $\Phi_{\mu}^{(1)}$'s in two sets $\Phi_{\mu'}^{(1)}$ and $\Phi_{\mu''}^{(1)}$ such that

$$\begin{aligned} \left\{ \Phi_{\mu'}^{(1)}, \Phi_{\nu}^{(1)} \right\} &\approx 0 \\ \left\{ \Phi_{\mu''}^{(1)}, \Phi_{\nu''}^{(1)} \right\} &\approx C_{\mu''\nu''}, \quad \det C_{\mu''\nu''} \neq 0. \end{aligned} \quad (18)$$

where the weak equality symbol \approx means equality on the surface of the constraints already known (here, the primary constraints). The matrix A can be decomposed to A' and A'' such that

$$\begin{aligned} A_{\mu'i} &= \partial_i \Phi_{\mu'}^{(1)} \\ A_{\mu''i} &= \partial_i \Phi_{\mu''}^{(1)}. \end{aligned} \quad (19)$$

Accordingly the symplectic tensor F can be written as

$$F = \left(\begin{array}{c|c|c} f & A'' & A' \\ \hline -\tilde{A}'' & 0 & 0 \\ \hline -\tilde{A}' & 0 & 0 \end{array} \right). \quad (20)$$

Consider the rectangular matrix

$$(\tilde{A}'f^{-1}, 0, 1) \quad (21)$$

which has M' rows and $2K + M$ columns. Using (18) one can show that its rows are left null-eigenvectors of F . Multiplying (21) with the equations of motion (15) gives the second level constraints as

$$\Phi_{\mu'}^{(2)} \approx \left\{ \Phi_{\mu'}^{(1)}, H \right\} = 0. \quad (22)$$

On the other hand, F in (20) has an invertible sub-block

$$F_{\text{inv}} = \left(\begin{array}{c|c} f & A'' \\ \hline -\tilde{A}'' & 0 \end{array} \right) \quad (23)$$

with the inverse

$$F_{\text{inv}}^{-1} = \left(\frac{f^{-1} - f^{-1} A'' C''^{-1} \tilde{A}'' f^{-1}}{C''^{-1} \tilde{A}'' f^{-1}} \middle| \frac{-f^{-1} A'' C''^{-1}}{C''^{-1}} \right). \quad (24)$$

This can solve the equations of motion (15) for variables $\dot{\eta}^{\mu''}$ to give

$$\dot{\eta}^{\mu''} = -C^{\mu''\nu''} \left\{ \Phi_{\nu''}^{(1)}, H \right\} \quad (25)$$

where $C^{\mu''\nu''}$ is the inverse of $C_{\mu''\nu''}$. Inserting this in the Lagrangian (12) gives

$$L^{(1)} = a_i(y) \dot{y}^i - \dot{\eta}^{\mu'} \Phi_{\mu'}^{(1)} - H^{(1)}(y) \quad (26)$$

where

$$H^{(1)} = H - \left\{ H, \Phi_{\mu''}^{(1)} \right\} C^{\mu''\nu''} \Phi_{\nu''}^{(1)}. \quad (27)$$

In this way a number of Lagrange multipliers corresponding to the second class constraints are derived whose effect is only replacing the canonical Hamiltonian H with $H^{(1)}$. Now we can forget about them and suppose that we are given the primary constraints $\Phi_{\mu}^{(1)}$ and the second level constraints $\phi_{\mu}^{(2)}$. Next, we should consider the consistency of $\Phi_{\mu}^{(2)}$ and add the term $-\dot{\eta}_2^{\mu} \Phi_{\mu}^{(2)}$ to the Lagrangian $L^{(1)}$. Renaming the previous $\eta^{\mu'}$'s as η_1^{μ} , the new Lagrangian would be

$$L^{(2)} = a_i(y) \dot{y}^i - \dot{\eta}_1^{\mu} \Phi_{\mu}^{(1)} - \dot{\eta}_2^{\mu} \Phi_{\mu}^{(2)} - H^{(1)}(y) \quad (28)$$

this gives the symplectic two-form

$$F^{(2)} = \left(\begin{array}{c|c|c} f & A^{(1)} & A^{(2)} \\ \hline -\tilde{A}^{(1)} & 0 & 0 \\ \hline -\tilde{A}^{(2)} & 0 & 0 \end{array} \right) \quad (29)$$

in the space (y, η_1, η_2) . Assuming that the composed matrix $A \equiv (A^{(0)}, A^{(1)})$, $F^{(2)}$ has the same form as (13). One should again proceed in the same way to find the null-eigenvectors as well as the invertible sub-block of $F^{(2)}$. The process goes on in this and the subsequent steps as explained in more detail in [10].

The important point to be emphasized is that the Lagrangian

$$L^{(n)} = a_i(y) \dot{y}^i - \sum_{k=1}^n \dot{\eta}_k^{\mu} \Phi_{\mu}^{(k)} - H^{(n)}(y) \quad (30)$$

at the n -th level, say, is equivalent to a system with extended Hamiltonian

$$H_E^{(n)} = H^{(n-1)} + \sum_{k=1}^n \lambda_k^\mu \Phi_\mu^{(k)} \quad (31)$$

at that level. In other words, the symplectic analysis is equivalent to the Dirac approach in the context of level by level method provided that at each level one adds the new constraints with the corresponding Lagrange multipliers to the Hamiltonian. In fact this slight difference with the standard Dirac method may lead to some difficulties as we will see in the following section.

3 The problem with extended Hamiltonian

The extended Hamiltonian formalism is well-known in the context of first class constraints [13, 14]. In fact, it can be shown that the dynamical equation

$$\dot{g} = \{g, H_E\}, \quad (32)$$

leads to the correct equation of motion provided that g is a gauge invariant quantity. In Eq. (32) the extended Hamiltonian H_E is defined as

$$H_E = H + \lambda^m \Phi_m \quad (33)$$

where Φ_m are only first class constraints (primary or secondary). For a first class system, the extended Hamiltonian can also be used step by step during the process of producing the constraints. In other words, when all of the constraints are first class, there is no difference whether one uses $\dot{\Phi} = \{\Phi, H_T\}$ or $\dot{\Phi} = \{\Phi, H_E\}$.

Now we show that the extended Hamiltonian formalism in Dirac approach is not suitable when second class constraints are present. We show this point for a system with only one primary constraint, i. e. a one-chain system in the language of chain by chain method. We remember that for such a system level by level and chain by chain methods coincide.

Consider a system with the canonical Hamiltonian $H(y)$ and one primary constraint $\Phi^{(1)}$. The total Hamiltonian reads

$$H_T = H + \lambda \Phi^{(1)}. \quad (34)$$

Suppose the consistency of $\Phi^{(1)}$ leads to $\Phi^{(2)} = \{\Phi^{(1)}, H\}$. Then $\Phi^{(3)}$ emerges as $\{\Phi^{(2)}, H\}$, and so on. The iterative process that produces the constraints is described by

$$\Phi^{(n+1)} = \{\Phi^{(n)}, H\}. \quad (35)$$

The above procedure progresses unless $\{\Phi^{(N)}, H_T\} \approx 0$ or $\{\Phi^{(N)}, \Phi^{(1)}\} \neq 0$ at the last step N . In the former case the constraints in the chain are first class, i.e. commute with each other [11]; while in the latter all the constraints are second class which means that the matrix

$$C^{nm} = \{\Phi^{(n)}, \Phi^{(m)}\} \quad (36)$$

is invertible. In this case the Lagrange multiplier λ would finally be determined as

$$\lambda = \frac{\{\Phi^{(N)}, H\}}{\{\Phi^{(N)}, \Phi^{(1)}\}}. \quad (37)$$

Using the Jacobi identity, it is shown in [11] that the matrix C^{nm} in Eq. (36) has the following form

$$C \approx \begin{pmatrix} 0 & 0 & \dots & 0 & C^{1N} \\ 0 & 0 & \dots & C^{2(N-1)} & C^{2N} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & C^{(N-1)2} & \dots & C^{(N-1)(N-1)} & C^{(N-1)N} \\ C^{N1} & C^{N2} & \dots & C^{N(N-1)} & C^{NN} \end{pmatrix}. \quad (38)$$

In other words

$$\{\Phi^{(i)}, \Phi^{(j)}\} \approx 0 \quad \text{if} \quad i + j \leq N. \quad (39)$$

Moreover using the Jacobi identity one can show from (35) that

$$\{\Phi^{(1)}, \Phi^{(N)}\} \approx -\{\Phi^{(2)}, \Phi^{(N-1)}\} \approx \dots \approx (-1)^{(\frac{N}{2}-1)} \{\Phi^{(\frac{N}{2})}, \Phi^{(\frac{N}{2}+1)}\} \neq 0. \quad (40)$$

Remember that N is the number of second class constraints and necessarily should be even.

Now suppose that in order to define the dynamics of the system at some level n , one wishes to use the extended Hamiltonian

$$H_E^{(n)} = H + \sum_{k=1}^n \lambda_k \Phi^{(k)}. \quad (41)$$

If $n \leq \frac{N}{2}$ then from (38) the consistency of the constraint $\Phi^{(n)}$ gives

$$\dot{\Phi}^{(n)} = \{\Phi^{(n)}, H_E^{(n+1)}\} \approx \{\Phi^{(n)}, H\} \quad (42)$$

which by (35), is the same as $\Phi^{(n+1)}$. However at level $\frac{N}{2} + 1$ the consistency of $\Phi^{(\frac{N}{2}+1)}$, using $H_E^{(\frac{N}{2}+1)}$ gives

$$\dot{\Phi}^{(\frac{N}{2}+1)} = \{\Phi^{(\frac{N}{2}+1)}, H\} + \lambda_{\frac{N}{2}} \{\Phi^{(\frac{N}{2}+1)}, \Phi^{(\frac{N}{2})}\}. \quad (43)$$

As is apparent from (40) the above equation solves the Lagrange multiplier $\lambda_{\frac{N}{2}}$. There is no justification to keep $\{\Phi^{(\frac{N}{2}+1)}, H\}$ as the next constraint $\Phi^{(\frac{N}{2}+2)}$. In order to knit the second class chain up to the last element $\Phi^{(N)}$, one is just allowed to use the total Hamiltonian (34). In other words, the second half of the chain can be derived if only the primary constraint $\Phi^{(1)}$ is present in the corresponding Hamiltonian. As explained in the previous section, using the standard symplectic analysis is equivalent to using the extended Hamiltonian formalism described above. So one should expect some contradiction in symplectic analysis when second class constraints are present. In the next section we will show the essence of this contradiction for a one chain system and propose a method to resolve it.

4 Second class one -chain in symplectic analysis

According to the algorithm given in section 2, given the canonical Hamiltonian $H(y)$ and the primary constraint $\Phi_\mu^{(1)}$, at the first step of consistency one should consider the Lagrangian (see 12)

$$L^{(1)} = a_i \dot{y}^i - \dot{\eta}_1 \Phi^{(1)} - H(y). \quad (44)$$

The equations of motion can be written in matrix form as

$$\left(\begin{array}{c|c} f & A^{(1)} \\ \hline -\tilde{A}^{(1)} & 0 \end{array} \right) \left(\begin{array}{c} \dot{y} \\ \dot{\eta}_1 \end{array} \right) = \left(\begin{array}{c} \partial H \\ 0 \end{array} \right). \quad (45)$$

It is easy to see that

$$u^1 \equiv (\tilde{A}^{(1)} f^{-1}, 1) \quad (46)$$

is the null-eigenvector of the matrix

$$F = \left(\begin{array}{c|c} f & A^{(1)} \\ \hline -\tilde{A}^{(1)} & 0 \end{array} \right). \quad (47)$$

Implying u^1 on both sides of (45) and using (5) gives the new constraint

$$\Phi^{(2)} = \{ \Phi^{(1)}, H \}. \quad (48)$$

Adding the term $-\dot{\eta}_2 \Phi^{(2)}$ to the Lagrangian (to perform consistency) gives

$$L^{(2)} = a_i \dot{y}^i - \dot{\eta}_1 \Phi^{(1)} - \dot{\eta}_2 \Phi^{(2)} - H(y). \quad (49)$$

The equations of motion are

$$\left(\begin{array}{c|c|c} f & A^{(1)} & A^{(2)} \\ \hline -\tilde{A}^{(1)} & 0 & 0 \\ \hline -\tilde{A}^{(2)} & 0 & 0 \end{array} \right) \begin{pmatrix} \dot{y} \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial y} \\ 0 \\ 0 \end{pmatrix} \quad (50)$$

Assuming $\{ \Phi^{(1)}, \Phi^{(2)} \} \approx 0$, one can find the new null eigenvector

$$u^2 \equiv (\tilde{A}^{(2)} f^{-1}, 0, 1). \quad (51)$$

Multiplying u^2 by (50) gives the new constraint $\Phi^{(3)} = \{ \Phi^{(2)}, H \}$, and so on.

Suppose one wishes to proceed in this way to find the constraints of the chain discussed in the previous section, i.e. the second class chain $\Phi^{(1)}, \dots, \Phi^{(N)}$ with the algebra given in (38-40). Suppose the above procedure has been proceeded up to the step $\frac{N}{2} + 1$ where the equations of motion are

$$\left(\begin{array}{c|c|c|c} f & A^{(1)} & \dots & A^{(\frac{N}{2}+1)} \\ \hline -\tilde{A}^{(1)} & 0 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline -\tilde{A}^{(\frac{N}{2}+1)} & 0 & \dots & 0 \end{array} \right) \begin{pmatrix} \dot{y} \\ \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_{\frac{N}{2}+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial y} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (52)$$

Clearly no more null-eigenvector can be find. In fact adding the column and row corresponding to the constraint $\Phi^{(\frac{N}{2}+1)}$ has increased the rank of the matrix F by two. This means that the equations of motion can be solved to

find $\dot{\eta}^{(\frac{N}{2})}$ and $\dot{\eta}^{(\frac{N}{2}+1)}$. There is no way in the context of symplectic analysis to proceed further to find the remaining constraints $\Phi^{(\frac{N}{2}+2)}, \dots, \Phi^{(N)}$ of the chain. This is really the failure of traditional symplectic analysis. In fact this is the reason why the symplectic analysis has failed in the example given in [12] (Particle in hyper sphere). We will discuss this example in section(6).

What we showed here is the failure of symplectic analysis for a second class system with only one primary constraint (i.e. a one chain system). However, one can easily observe that for an arbitrary system with several primary constraints again the symplectic analysis would fail. The reason is that for such a system some of the constraints driven at level n , i.e. $\Phi_\mu^{(n)}$, may have non vanishing Poisson brackets with constraints of previous levels while commuting with primary constraints. As we know from Dirac approach, in such a case the Poisson brackets of these constraints with Hamiltonian give the next level constraints. Meanwhile, a little care on symplectic analysis shows that in this case a number of Lagrange Multipliers corresponding to non-primary constraints would be determined and there is no way to find the next level constraints. In this way, we conclude that *the symplectic analysis would fail whenever second class constraints emerge at third level or higher.*

5 How to solve the problem

In this section we try to find a way to maintain the symplectic analysis by imposing some modifications. The origin of the problem is the fact that $\Phi^{(\frac{N}{2}+1)}$ has non-vanishing Poisson bracket with $\Phi^{(\frac{N}{2})}$. As a result, the symplectic two-form on the left hand side of Eq. (52), i.e.

$$F = \left(\begin{array}{c|c|c|c} f & A^{(1)} & \dots & A^{(\frac{N}{2}+1)} \\ \hline -\tilde{A}^{(1)} & 0 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline -\tilde{A}^{(\frac{N}{2}+1)} & 0 & \dots & 0 \end{array} \right), \quad (53)$$

does not possess a new null-eigenvector. If one could consider the vector

$$u^{(\frac{N}{2}+1)} \equiv \left(\tilde{A}^{(\frac{N}{2}+1)} f^{-1}, 0, \dots, 0, 1 \right), \quad (54)$$

as a null-eigenvector, then by multiplying $u^{(\frac{N}{2}+1)}$ on the right hand side of (52), one would obtain the next constraint as

$$\Phi^{(\frac{N}{2}+2)} = \left\{ \Phi^{(\frac{N}{2}+1)}, H \right\}. \quad (55)$$

To reach this goal one should truncate those columns of F which are located after $A^{(1)}$. In other words, instead of F in Eq. (53) one should consider the rectangular matrix

$$\tilde{F} = \left(\begin{array}{c|c} f & A^{(1)} \\ \hline -\tilde{A}^{(1)} & 0 \\ \hline \vdots & \vdots \\ \hline -\tilde{A}^{(\frac{N}{2}+1)} & 0 \end{array} \right). \quad (56)$$

Clearly $u^{(\frac{N}{2}+1)}$ in Eq. (54) is the null-eigenvector of \tilde{F} . It is obvious that if one does the same thing in the subsequent steps, one can produce all the remaining constraints of the chain, i.e. $\Phi^{(\frac{N}{2}+1)}, \dots, \Phi^{(N)}$. In the last step the chain terminates, since $\left\{ \Phi^{(N)}, \Phi^{(1)} \right\} \neq 0$.

But what is the justification to find the null-eigenvectors of \tilde{F} , i.e. the *truncated* F . In fact using Eq. (5) the set of equations

$$\left(\begin{array}{c|c} f & A^{(1)} \\ \hline -\tilde{A}^{(1)} & 0 \\ \hline \vdots & \vdots \\ \hline -\tilde{A}^{(N)} & 0 \end{array} \right) \left(\begin{array}{c} \dot{y} \\ \dot{\eta}_1 \end{array} \right) = \left(\begin{array}{c} \frac{\partial H}{\partial y} \\ 0 \\ \vdots \\ 0 \end{array} \right). \quad (57)$$

is equivalent to

$$\begin{aligned} \dot{y}_i &= \left\{ y_i, H + \dot{\eta}_1 \Phi^{(1)} \right\} & i &= 1, \dots, 2K \\ \dot{\Phi}^{(j)} &= 0 & j &= 1, \dots, N. \end{aligned} \quad (58)$$

Remembering that $\dot{\eta}_1$ has the same role as the Lagrange multiplier λ_1 corresponding to the primary constraint $\Phi^{(1)}$, we see that Eq. (58) is the correct equation of motion

$$\dot{y}_i = \{y_i, H_T\}. \quad (59)$$

On the other hand, it is easy to see that the equations of motion resulting from Eq. (52) can be written as

$$\dot{y}_i = \{y_i, H_E\} \quad (60)$$

where H_E contains all derived constraint(including second class ones). In fact as we explained before, the correct equations of motion are (58) and not (60).

Therefore, if one wishes to proceed in the context of symplectic analysis, one should consider Eq.(57) instead of Eq.(52)

6 Example

Consider the Lagrangian

$$L = \frac{1}{2}\dot{\mathbf{q}}^2 + v(\mathbf{q}^2 - 1) \quad (61)$$

where $\mathbf{q} \equiv (q_1, \dots, q_n)$. The primary constraint is P_v . The corresponding Hamiltonian is

$$H = \frac{1}{2}\mathbf{p}^2 - v(\mathbf{q}^2 - 1) \quad (62)$$

where $\mathbf{p} \equiv (p_1, \dots, p_n)$. In the usual Dirac approach, using the total Hamiltonian $H_T = H + \lambda P_v$, the consistency of $\Phi^{(1)} = P_v$ gives the following chain of constraints

$$\begin{aligned} \Phi^{(1)} &= P_v \\ \Phi^{(2)} &= \mathbf{q}^2 - 1 \\ \Phi^{(3)} &= 2\mathbf{q} \cdot \mathbf{p} \\ \Phi^{(4)} &= 2(\mathbf{p}^2 + 2v\mathbf{q}^2) \end{aligned} \quad (63)$$

As is apparent, $\Phi^{(4)}$ and $\Phi^{(3)}$ are conjugate to $\Phi^{(1)}$ and $\Phi^{(2)}$ respectively. It is worth remembering that although $\Phi^{(3)}$ is second class, when reaching at third level, the process of consistency should not stop, i. e. it should be proceeded one level more to find $\Phi^{(4)}$ which is conjugate to the primary constraint $\Phi^{(1)}$. In the symplectic approach the corresponding first order Lagrangian is

$$L = \mathbf{p}\dot{\mathbf{q}} + P_v\dot{v} - \frac{1}{2}\mathbf{p}^2 + v(\mathbf{q}^2 - 1) - \lambda P_v. \quad (64)$$

This gives the singular presymplectic tensor

$$F = \left(\begin{array}{c|c} f & 0 \\ \hline 0 & 0 \end{array} \right) \quad (65)$$

where f is the usual $(2n+2) \times (2n+2)$ symplectic tensor:

$$f = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array} \right). \quad (66)$$

The equations of motion for $y^i = (\mathbf{q}, v, \mathbf{p}, P_v, \lambda)$ are $f_{ij}\dot{y}^j = \partial_i H_T$ where $H_T = H + \lambda P_v$. Clearly this gives the canonical equation of motion with Hamiltonian H_T , together with the constraint equation $P_v = 0$. Adding the consistency term $-\dot{\eta}_1 P_v$ to the Lagrangian (64), where η_1 is a new variable and forgetting about the term proportional to λ (which just reproduces the primary constraint) one finds

$$L^{(1)} = \mathbf{p}\dot{\mathbf{q}} + P_v \dot{v} - \dot{\eta}_1 P_v - \frac{1}{2}\mathbf{p}^2 + v(\mathbf{q}^2 - 1). \quad (67)$$

This gives the equations of motion

$$F_{ij}^{(1)}\dot{Y}^j = \partial_i H \quad (68)$$

where $Y^i \equiv (\mathbf{q}, v, \mathbf{p}, P_v, \eta_1)$. In the matrix form we have

$$F^{(1)} = \left(\begin{array}{c|c} f & A^{(1)} \\ \hline -A^{(1)} & 0 \end{array} \right) \quad (69)$$

where $\tilde{A}^{(1)} = (\mathbf{0}, 0, \mathbf{0}, 1)$. Here, bold zero ($\mathbf{0}$) means a row vector with n zero components. Clearly $u^{(1)} = (\mathbf{0}, -1, \mathbf{0}, 0, 1)$ is the left null-eigenvector of $F^{(1)}$. Multiplying the equations of motion (68) from the left by $u^{(1)}$ gives the constraint $\Phi^{(2)} = \mathbf{q}^2 - 1$.

In the next level we have the Lagrangian

$$L^{(2)} = L - \dot{\eta}_1 P_v - \dot{\eta}_2 (\mathbf{q}^2 - 1) \quad (70)$$

written in the space $Y^i \equiv (\mathbf{q}, v, \mathbf{p}, P_v, \eta_1, \eta_2)$. The corresponding symplectic tensor reads

$$F^{(2)} = \left(\begin{array}{c|cc} f & A^{(1)} & A^{(2)} \\ \hline -\tilde{A}^{(1)} & 0 & 0 \\ -\tilde{A}^{(2)} & 0 & 0 \end{array} \right) \quad (71)$$

where $\tilde{A}^{(2)} = (2\mathbf{q}, 0, \mathbf{0}, 0)$. Clearly $u^{(2)} = (\mathbf{0}, 0, 2\mathbf{q}, 0, 0, 1)$ is the null-eigenvector of $F^{(2)}$. Multiplying the equations of motion $F_{ij}^{(2)}\dot{Y}^j = \partial_i H_T$ from the left by

$u^{(2)}$ gives the next level constraint $\Phi^{(3)} = 2\mathbf{q} \cdot \mathbf{p}$. Again considering another variable η_3 , the third level Lagrangian would be

$$L^{(3)} = L - \dot{\eta}_1 P_v - \dot{\eta}_2 (\mathbf{q}^2 - 1) - \dot{\eta}_3 (2\mathbf{q} \cdot \mathbf{p}). \quad (72)$$

This gives the following symplectic tensor

$$F^{(3)} = \left(\begin{array}{c|ccc} f & A^{(1)} & A^{(2)} & A^{(3)} \\ \hline -\tilde{A}^{(1)} & 0 & 0 & 0 \\ -\tilde{A}^{(2)} & 0 & 0 & 0 \\ -\tilde{A}^{(3)} & 0 & 0 & 0 \end{array} \right) \quad (73)$$

where $\tilde{A}^{(3)} = (2\mathbf{p}, 0, 2\mathbf{q}, 0)$. Now the crucial point appears. That is, $F^{(3)}$ has no new null-eigenvector. In fact one expects that multiplying $u^{(3)} = (-2\mathbf{q}, 0, 2\mathbf{p}, 0, 0, 0, 1)$ by the equations of motion due to $L^{(3)}$ gives the next constraint $\Phi^{(4)} = 2(\mathbf{p}^2 + 2v\mathbf{q}^2)$. However, it can be easily checked that $u^{(3)} F^{(3)} \neq 0$. Moreover, $u^{(2)}$ (with one additional zero as the last element) is no more the null-eigenvector of $F^{(3)}$. This means that adding the $(2n+5)$ th row and columns to $F^{(2)}$ has led to increasing the rank of $F^{(3)}$ by two. In other words, the equations of motion for $\dot{\eta}_2$ and $\dot{\eta}_3$ can be solved. Unfortunately without any modification **there is no way to find** the Lagrangian

$$L^{(4)} = L - \dot{\eta}_1 P_v - \dot{\eta}_2 (\mathbf{q}^2 - 1) - \dot{\eta}_3 (2\mathbf{q} \cdot \mathbf{p}) - \dot{\eta}_4 (2(\mathbf{p}^2 + 2v\mathbf{q}^2)). \quad (74)$$

If we could find $L^{(4)}$, then we would be able to have

$$F^{(4)} = \left(\begin{array}{c|cccc} f & A^{(1)} & A^{(2)} & A^{(3)} & A^{(4)} \\ \hline -\tilde{A}^{(1)} & 0 & 0 & 0 & 0 \\ -\tilde{A}^{(2)} & 0 & 0 & 0 & 0 \\ -\tilde{A}^{(3)} & 0 & 0 & 0 & 0 \\ -\tilde{A}^{(4)} & 0 & 0 & 0 & 0 \end{array} \right) \quad (75)$$

where $\tilde{A}^{(4)} = (8v\mathbf{q}, 4\mathbf{q}^2, 4\mathbf{p}, 0)$. If we had somehow derived (74) and (75), then the singularity of symplectic tensor would completely disappear and $\dot{\eta}_1, \dots, \dot{\eta}_4$ would be obtained. However, using the *truncated symplectic tensor* at the second step as

$$\tilde{F}^{(2)} = \left(\begin{array}{c|c} f & A^{(1)} \\ \hline -\tilde{A}^{(1)} & 0 \\ -\tilde{A}^{(2)} & 0 \end{array} \right) \quad (76)$$

and similarly $\tilde{F}^{(3)}$ at the third level as

$$\tilde{F}^{(3)} = \left(\begin{array}{c|c} f & A^{(1)} \\ \hline -\tilde{A}^{(1)} & 0 \\ -\tilde{A}^{(2)} & 0 \\ -\tilde{A}^{(3)} & 0 \end{array} \right) \quad (77)$$

makes it possible to introduce again $u^{(2)}$ and $u^{(3)}$ as the corresponding left null-eigenvectors of $\tilde{F}^{(2)}$ and $\tilde{F}^{(3)}$, respectively. This makes us able to find $\Phi^{(4)}$ as explained before. It should be noted that one can after all write the complete symplectic tensor $F^{(4)}$.

This example has also been discussed in [12], where some other reason is proposed as the origin of failure of the symplectic analysis. The same results as what we derived here can be found in every second class system possessing at least four levels of constraints. For example, one can study the simpler Lagrangian $L = \dot{x}\dot{y} - z(x + y)$ as well as the more complicated example of bosonized Schwinger model in $(1 + 1)$ dimensions [15, 16] given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (g^{\mu\nu} - \varepsilon^{\mu\nu})\partial_\mu\phi A_\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu A^\mu. \quad (78)$$

One can see that the main feature of the above calculations will more or less appear in all such examples.

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References

- [1] M. Gotay, J. M. Nester and G. Hinds, J. Math. Phys. 19, 2388(1978)
- [2] S. Hojman and F. Urrutia, J. Math. Phys. 22, 1896(1981)
- [3] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60, 1692(1988).
- [4] J. A. Garcia and J. M. Pons, Int. J. Mod. Phys. A12, 451(1997).
- [5] P. A. M. Dirac, "Lectures on Quantum Mechanics" New York: Yeshiva University Press (1964).
- [6] J. Barcelos-Neto and C. Wotzasck, Mod. Phys. Lett. A7, 1737(1992).
- [7] J. Barcelos-Neto and C. Wotzasck, Int. Mod. Phys. Lett. A7, 4981(1992).

- [8] J. Barcelos-Neto and N. R. F. Beraga, J. Math. Phys. 35, No. 7, 3497(1994).
- [9] H. Montani Int. Mod. Phys. A8, 4319(1993).
- [10] A. Shirzad and M. Mojiri, Mod. Phys. Lett. A16, No. 38, 2439(2001).
- [11] F. Loran and A. Shirzad, Int. J. Mod. Phys. A17, 625(2002).
- [12] H. J. Rothe and K. D. Rothe, J. Phys. A36, 1671(2003).
- [13] M. Henneaux and C. Teitelboim, *"Quantization of Gauge Systems"*, Princeton Univ. Press (1992).
- [14] C. Batlle, J. M. Pons and N. Roman-Roy, J. Math. Phys. 27, 2953(1986).
- [15] P. Mitra and R. Rajaraman, Phys. Lett. B225, 267(1989).
- [16] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54, 1219(1985).
- [17] J. Saavedra, R. Troncoso and J. Zanelli, J. Math. Phys. 42, 4383(2001).
- [18] O. Miskovic and J. Zanelli, J. Math. Phys. 44, 3876(2003).